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Phil. Trans. R. Soc. Lond. A 1976 **281**, 97-111
doi: 10.1098/rsta.1976.0022

Phil. Trans. R. Soc. Lond. A. **281**, 97–111 (1976) [97]
Printed in Great Britain

A THEORY OF SHORT PERIOD TIDES IN A ROTATING BASIN

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(*Communicated by C. H. Mortimer, F.R.S. – Received* 21 *June* 1974)

CONTENTS

An analytical model of the forced tides in a rotating and enclosed basin of variable depth is formulated which includes the effects of basin shape, the rotating Froude number, latitude, angle of orientation and dissipation on the semidiurnal and diurnal response. To a crude approximation *contra solem* rotation of nodal lines of the semidiurnal tide and diurnal tide at latitudes north at 45° occurs when the frequency of tidal forcing falls between the frequencies of the transverse and longitudinal free modes of oscillation. Predicted phase responses for the semidiurnal tide agree with measurements in the seven basins studied. The anomolous response found in Lake Michigan by Mortimer & Fee is attributed to bottom friction coincidence with an appropriate angle of orientation of the basin. Amplitude response in all cases is larger than observed which cannot be accounted for generally by reasonable coefficients of friction.

1. Introduction

In an intensive study of the fluctuations of water levels of Lake Superior and Lake Michigan, Mortimer & Fee (1976) have observed the lunar semidiurnal tidal response of these Great Lakes. Mortimer & Fee's finding that the directions of rotation of the nodal lines associated with this tidal constituent are opposite in the two lakes of approximately the same size and latitude is curious and difficult to account for on the basis of classical theory. Their result has motivated a re-examination of this mostly Victorian problem of tidal generation in an enclosed basin.

Although digital techniques have revolutionized the subject of tidal theory in recent years, numerical calculations at times are as difficult to interpret as the observed phenomenon (Csanady 1973). Therefore, in order to gain an understanding of the dynamics of the observed tidal response an analytical model is proposed of sufficient simplicity to exhibit the basic observable features involved and yet comprehensive enough to include the fundamental determining influences of rotation, lateral boundaries, variable depth and dissipation.

The present paper is concerned with the oscillations of the mass of water contained in a region which is acted upon by the tidal generating forces. The basin is of sufficiently large extent that rotation is an important factor in the dynamics but not so large that the variation of the vertical component of the Earth's rotation need be accounted for. Because the sense of rotation of the forced wave is an easily observable qualitative feature of the tidal response of an enclosed basin, attention is focused throughout the discussion on the influence of the various determining factors on the direction of rotation of the current field and of the free surface. Theoretical predictions of the nature of the semidiurnal tide are compared to Mortimer & Fee's observations as well as an analysis of water level data in Lake Huron and in Lake Ontario. In the latter study data were collected during a detailed observational programme known as the International Field Year for the Great Lakes.

2. The horizontal components of the tidal generating forces

For the purpose of illustrating the theory the responses of the principal diurnal species, K_1, and the principal semidiurnal species, M_2, are studied. Following Hendershott & Munk (1970) the equilibrium tide, $\bar{z}(t)$, can be written in the form

$$\bar{z}(t) = CKP_n^m(\theta)\cos(\sigma t + n\phi + \alpha k)$$

at a latitude, θ, longitude, ϕ, and frequency, σ. K is a constant equal to 53.7 cm and n, C and αK and P_n^m are specified for the individual tidal constituents.

The associated horizontal components of tidal force per unit volume of water are, in the east direction, F'_x,

$$F'_x = \frac{g\rho}{r\cos\theta}\frac{\partial\bar{z}}{\partial\phi}$$

and in the north direction, F'_y,

$$F'_y = \frac{\rho g}{r}\frac{\partial\bar{z}}{\partial\theta},$$

where r is the mean spherical radius of the Earth, g the acceleration due to gravity and ρ, the density of the fluid.

For the semidiurnal constituent, M_2, the horizontal force is expressed as,

$$\left.\begin{aligned} F'_x &= -\rho K(0.908/r)\,g\cos\theta\sin(\sigma t + 2\phi + \alpha M_2),\\ F'_y &= -\rho K(0.908/r)\,g\cos\theta\sin\theta\cos(\sigma t + 2\phi + \alpha M_2),\end{aligned}\right\} \tag{1}$$

where the period of forcing is 12.42 h.

Similarly the components of horizontal force are, for the K_1 diurnal species of period 23.93 h,

$$\left.\begin{aligned} F'_x &= -\rho K(0.531/r)\,g\sin\theta\sin(\sigma t + \phi + \alpha K_1),\\ F'_y &= -\rho K(0.531/r)\,g[\sin^2\theta - \cos^2\theta]\cos(\sigma t + \phi + \alpha K_1).\end{aligned}\right\} \tag{2}$$

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As is well known from tidal theory, the semidiurnal force rotates in a clockwise sense for all positive latitudes whereas the diurnal force rotates clockwise north of 45° N and counterclockwise south of 45° N.

3. The model equations

Consider a basin of elliptical plan and parabolic depth profile whose major axis is orientated at an angle of γ to a parallel of latitude as illustrated schematically in figure 1. If a local Cartesian coordinate system is defined (x, y) such that the x axis lies along the major axis of the ellipse then the classical linearized Laplace tidal equations appropriate for a plane of constant rotation reduce to

$$\frac{\partial u}{\partial t}-fv = -g\frac{\partial \eta}{\partial x}+\frac{Fx}{\rho}+\frac{Fbx}{h\rho},$$

$$\frac{\partial v}{\partial t}+fu = -g\frac{\partial \eta}{\partial y}+\frac{Fy}{\rho}+\frac{Fby}{h\rho},$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y} = 0,$$

where f is the Coriolis parameter, and (u, v) are the x and y components of velocity in the local coordinate system. In addition, η is the free surface displacement from its undisturbed level.

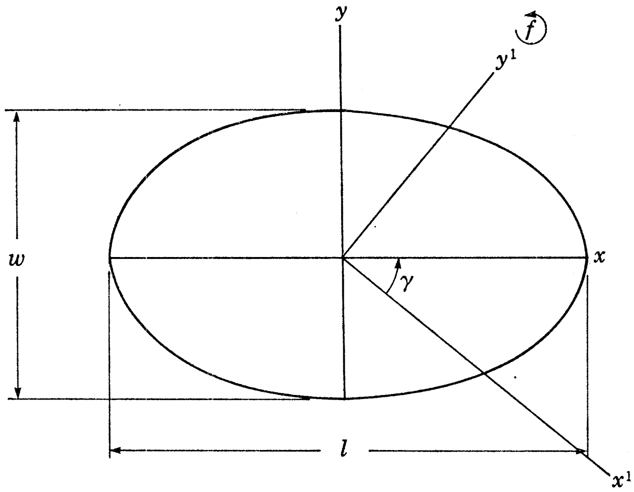

Figure 1. Schematic diagram of model basin in plan.

The undisturbed depth, h, varies according to the law

$$h(x,y) = H_0-\tfrac{1}{2}\alpha x^2-\tfrac{1}{2}\beta y^2,$$

where the depth in the centre of the basin is H_0. If the major axis has a length, l, and the minor axis a width, W, then

$$\alpha = 8H_0/l^2; \quad \beta = 8H_0/W^2.$$

The quantities F_{bx} and F_{by} are the components of bottom stress which will be specified at a latter stage. The tidal forces in the local coordinate system are

$$F_x = F'_x \cos\gamma + F'_y \sin\gamma,$$

and

$$F_y = F'_y \cos\gamma - F'_x \sin\gamma.$$

The lateral boundary condition requires that the current components remain finite as the depth approaches zero at the shoreline of the basin. There are, of course, no temporal conditions other than that the solution be periodic in time.

4. The Inviscid model

(a) *Formulation of the model*

As the point of departure for the study, the behaviour of the model will first be examined without the complication of bottom friction. Since there are a considerable number of parameters in the model for example, g, f, H_0, l, and W it is possible to simplify the analysis if the Laplace tidal equations are non-dimensionalized. Non-dimensional variables in the local coordinate system will be denoted by the primed notation.

The variables in the equations of motion are scaled by a velocity, Lf, where the horizontal basin scale is L, depth, H_0, time, f^{-1}, and tidal force per unit mass gH_0/L. Vertical displacement is scaled by H_0. Primes now denote non-dimensional variables.

Specifically, the depth law is scaled as

$$h = H_0 h' = H_0[1-(\alpha/2H_0)(x'L)^2-(\beta/2H_0)(y'L)^2],$$

if

$$\alpha L^2/H_0 = 1-a \quad \text{and} \quad \beta L^2/H_0 = 1+a,$$

then

$$h' = 1-\tfrac{1}{2}(1-a)x'^2-\tfrac{1}{2}(1+a)y'^2,$$

where it follows that, $a = (\beta-\alpha)/(\alpha+\beta)$, is a shape factor associated with the basin which varies from zero for circular plan to unity as the basin approaches an infinitely long channel. A further consequence of this scaling is that the basin scale, L, is of the form,

$$L = \frac{1}{2}\sqrt{\frac{W^2l^2}{l^2+W^2}}.$$

The model equations become, upon non-dimensionalization and dropping the primes

$$\frac{\partial u}{\partial t}-v = -\epsilon\frac{\partial \eta}{\partial x}+\epsilon F_x,$$

$$\frac{\partial v}{\partial t}-u = -\epsilon\frac{\partial \eta}{\partial y}+\epsilon F_y,$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y} = 0.$$

For convenience, the non-dimensional parameter, ϵ, will be referred to as the inverse rotating Froude number:

$$\epsilon = gH_0/f^2L^2.$$

It is evident from the expressions (1) and (2), for the tidal generating forces, appropriate to a basin of east–west orientation that F_x varies as $\sin \omega t$ and F_y as $\cos \omega t$ where ω is the non-dimensional frequency of oscillation, $\sigma = \omega f$.

The periodic solution satisfying the boundary conditions may be expressed for a basin orientated in an east–west direction, as

$$u = U\cos \omega t,$$

$$v = V\sin \omega t,$$

$$\eta = Ax\sin \omega t+By\cos \omega t,$$

The coefficients of the planar free surface, A and B, and the coefficients of the spatially uniform velocity field, U and V, may be determined by substitution into the inviscid equations of motion

$$-\omega U - V = -\epsilon A + \epsilon F_x, \tag{3}$$

$$\omega V + U = -\epsilon B + \epsilon F_y, \tag{4}$$

$$\omega A - (1-a)\, U = 0, \tag{5}$$

$$-\omega B - (1+a)\, V = 0. \tag{6}$$

Based on elimination of A from equations (3) and (5)

$$[-\omega + (1-a)\,\epsilon/\omega]\, U - V = \epsilon F_x. \tag{7}$$

Similarly, from equations (4) and (6)

$$U + [\omega - \epsilon(1+a)/\omega]\, V = \epsilon F_y, \tag{8}$$

Finally, from equations (7) and (8), the following solutions result:

$$U = -\frac{[\omega^2 - \epsilon(1+a)]\,\epsilon\omega F_x + \epsilon\omega^2 F_y}{(\omega^2 - \omega_1^2)\,(\omega^2 - \omega_t^2)}, \tag{9}$$

$$V = \frac{[\omega^2 - (1-a)\,\epsilon]\,\omega\epsilon F_y + \epsilon\omega^2 F_x}{(\omega^2 - \omega_1^2)\,(\omega^2 - \omega_t^2)}. \tag{10}$$

A and B follow from equations (5) and (6).

In expressions (9) and (10), ω_1 is the lowest longitudinal frequency of free oscillation. Similarly ω_t is the fundamental transverse frequency of oscillation. These frequencies are the roots, λ_1 and λ_2, of the equation

$$\lambda^4 - (2\epsilon + 1)\,\lambda^2 + \epsilon^2(1-a^2) = 0,$$

which is of the same form as the denominator of equations (9) and (10).

If $\qquad \omega_1^2 \quad \text{and} \quad \omega_t^2 \gg 1,$

then $\qquad \omega_1^2 \approx \epsilon(1-a),$

and $\qquad \omega_t^2 \approx \epsilon(1+a).$

The approximation becomes an equality if rotation is absent.

(b) *Discussion*

The flow vector whose components are u and v rotates in a *contra solem* direction if U and V are both of similar sign and in the opposite sense if the signs are reversed.

If the forcing frequency is very much less than the frequency of the fundamental mode of oscillation so that

$$\omega^2 \ll w_1^2 < \omega_t^2,$$

then (9) and (10) to a first approximation become

$$U \cong \frac{\epsilon\omega\omega_t F_x - \epsilon\omega^2 F_y}{\omega^2(1-a^2)},$$

and

$$V \cong \frac{-\epsilon\omega_1\omega F_y + \epsilon\omega^2 F_x}{\omega^2(1-a^2)}.$$

If F_x and F_y are of the same sign and F_y is not a great deal less than F_x as is the case for the semi-diurnal tide and diurnal tide at latitudes higher than 45°, it is apparent from the above expressions that the signs of U and V are opposite. Thus, the response is in the direction of forcing. A similar situation exists for the diurnal response at latitudes less than 45° N. It is concluded that for a basin of appropriately high frequencies of natural oscillation the forced response assumes the form of a quasi-static directly forced mode.

If we consider the situation as the horizontal basin dimensions increase or as the depth decreases, either of which result in a decrease of the associated frequencies of free oscillation, a point is reached at which the east component of current amplitude is equal to zero.

In this case, expression (10) yields values of ϵ and a, ϵ_1 and a_1 such that

$$\omega = \dagger F_x/2F_y + \tfrac{1}{2}\sqrt{[(F_x/F_y)^2 + 4(1-a_1)\,\epsilon_1]}.$$

Continuing this process of decreasing the natural frequencies of oscillation, U eventually becomes zero. At this point the parameters a and ϵ are denoted by a_2, ϵ_2

$$\omega = -F_y/2F_x + \tfrac{1}{2}\sqrt{[(F_y/F_x)^2 + 4(1+a_2)\,\epsilon_2]}.$$

The region of basin scale sizes of the basin lying between the limits ϵ_1, a_1 and ϵ_2, a_2 may be represented in frequency by the approximate relation $\omega_1 < \omega < \omega_t$.

Reference to equations (9) and (10) indicates that, due to a similarity of sign, the phase response is in the cyclonic sense when the frequency of longitudinal oscillation is less than the forcing frequency, which in turn is less than the transverse frequency of oscillation. The response in this region may readily be identified with that of the longitudinal free mode of oscillation of the basin. If the right-hand sides of equations (7) and (8) are set to zero as is the case for the unforced solutions, it follows that U and V are of the same sign when $\omega^2 \simeq \epsilon(1-a)$ in equation (8). The cyclonic response of the tidal oscillation is attributed to the predominant excitation of the longitudinal mode under these conditions.

Finally, by means of a similar argument it can be shown that the motion is anti-cyclonic if $\omega_t < \omega$. It is evident from equation (7) that when $F_x = 0$ and $\omega \simeq \epsilon(1+a) > \epsilon(1-a)$, U and V are of opposite sign. Thus, the response in this situation is approximately represented by the forced transverse mode of oscillation.

These concepts are illustrated in a more precise manner in figures 2 and 3. Specifically, ϵ_1 and ϵ_2 are plotted as a function of the parameter, a, for the two cases of east–west and north–south orientation of the major axis of the basin. For reference, the locus of the transverse and longitudinal oscillations having the same frequency as the forcing frequency are depicted. Other graphical displays similar to figure 2 indicate a fairly weak dependence of the semidiurnal response curves with latitude over a range from 30° to 75° N. Therefore, the response curves at 45° may be considered as representative of the behaviour of the semidiurnal tide at mid-latitudes.

The approximate correspondence between the response curves and the resonance curves especially for more elongated basins, confirms the earlier suggestion that the direction of semi-diurnal response can be estimated on the basis of knowledge of the fundamental periods of oscillation of the basin. The response curves subdivide the parameter space into three regions, an upper region where the quasi-static mode predominates, an intermediate zone where longitudinal response is excited, and a lower portion where transverse response is generated.

† From expression (1) $F_x/F_y = 1/\sin\theta$ for east–west orientation and $\sin\theta$ for north–south orientation of the major axis of the basin.

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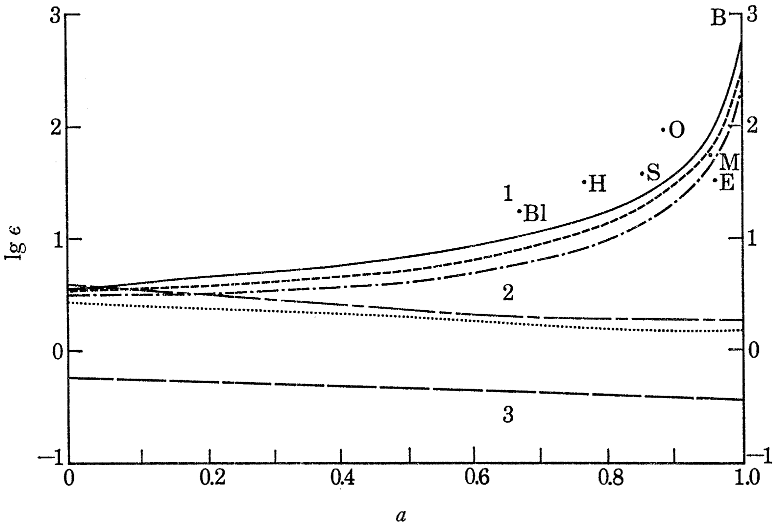

FIGURE 2. Semidiurnal response curves as a function of shape factor, a, and rotating Froude number, ϵ, at 45° N latitude. Regions 1 and 3 denote *cum sole* response while region 2 is of the opposite sense. ——, ϵ_1 east orientation; ------, ϵ_1 north; —·—·, longitudinal period of free oscillation equal to the semi-diurnal period; ……, ϵ_2 east; ——–-, ϵ_2 north; — —, transverse period of free oscillation equal to the semi-diurnal period; E, Lake Erie; S, Lake Superior; Bl, Black Sea; B, Baikal; H, Huron; O, Ontario; M, Michigan.

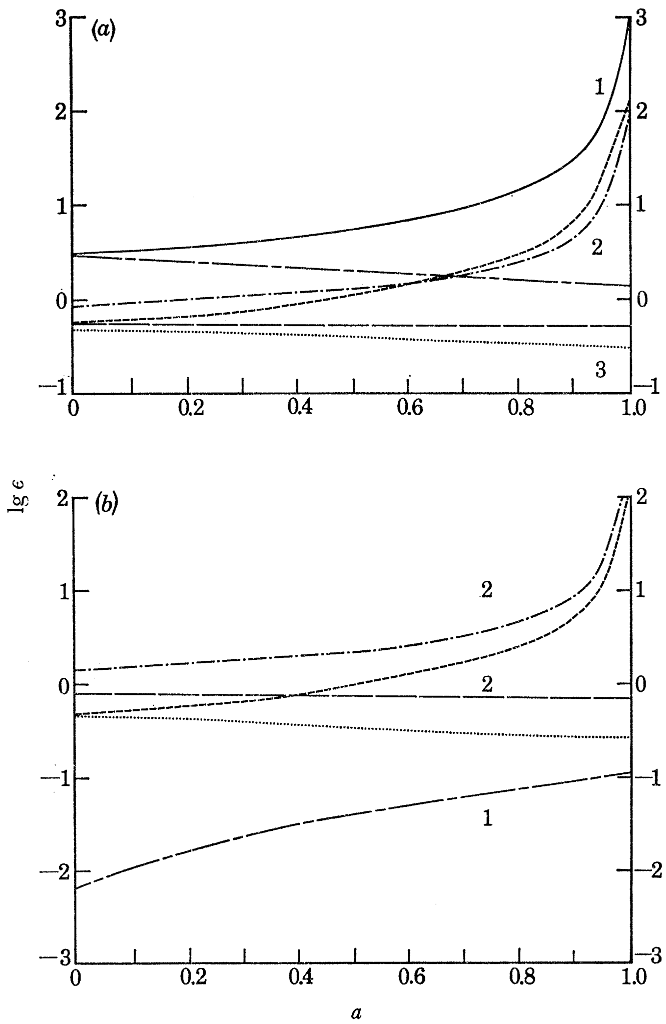

FIGURE 3. Diurnal response curves as a function of shape factor, a, and rotating Froude number, ϵ, at (a), 50 °N. latitude, (b) 40° N latitude. Region 1 and 3 denote *cum sole* response while region 2 is of the opposite sense ——, ϵ_1 east orientation; ------, ϵ_1 north; —·—·, longitudinal period of free oscillation equal to the diurnal period; ……, ϵ_2 east; ——–-, ϵ_2 north orientation; — —, transverse period of free oscillation equal to the diurnal period. In (b), ϵ_1 east is less than −2.0 and as a result is not plotted.

As the basin becomes progressively more elongated, *contra solem* response obtains over a wider range of rotating Froude numbers. There is weakly demonstrated tendency for more pronounced longitudinal response for basins orientated in zonal directions than for basins orientated in meridional directions. In fact, for a basin of north–south alinement and nearly circular plan ($a = 0.15$) cyclonic response is not possible for any rotating Froude number.

A more complex situation exists for the diurnally forced tide. In this case the direction of rotation of the forcing function changes at 45° N latitude. North of this latitude, for example at 50°, the response is similar to the semidiurnal situation. Curves dividing $\epsilon - a$ space into three regions of direct and inverted response are depicted in figure 3*a* for basins of zonal and meridional alinement. The influence of basin direction is more pronounced than in the semidiurnal case due to the greater ellipticity of the forcing function. It is noteworthy that for a basin of meridional extension the elongation ($a = 0.7$) at which no inverse response occurs is much more substantial than is the case for semidiurnal forcing.

Table 1. Data on basin geometry for seven enclosed basins. Symbols are defined in the text

basin	area km²	volume km³	H_0 m	long period h	effective length km	effective width km	latitude deg	angle of orien-tation deg	l km	ϵ	a
Superior	78 500†	11 550†	294	7.89‡	686	145	47.5	0	71	50	0.91
Michigan	52 500†	4640†	178	9.0‡	620	108	43.5	55	53	60	0.94
Huron	40 800†	2 660†	130	6.5†	375†	138	44.5	−50	65	29	0.76
Erie	25 400†	480†	38	14.1†	440†	74	42.0	30	36	29	0.94
Ontario	18 400†	1 580†	172	5.0†	333	70	43.5	10	34	141	0.92
Baikal	31 500§	23 000§	1480	4.85§	950	42	53.0	50	21	2240	0.996
Black Sea	$4.61\| \times 10^5$	$5.37 \times 10^5\|$	2332	5.0¶	1230	477	43.0	0	222	46	0.74

† Rockwell (1966). ‡ Mortimer & Fee (1976). § Kozhov (1963).
|| Handbook of Oceanographic Tables 1966. US Naval Ocn. Office, Washington, D.C.
¶ Defant (1961).

In contrast to the sample basin situated at 50° N, for the basin located at 40° N, figure 3*b*, the direct response portion of the parameter space represents clockwise motion. Because of the extremely low rotating Froude numbers involved, longitudinal or transverse response is unlikely to occur for natural basins with the possible exception of extremely elongated basins.

An assemblage of ϵ, a coordinates derived from the geographical data of table 1 for a number of basins of known semidiurnal response are plotted in figure 2. With the possible exception of Lake Michigan all coordinates are located in the appropriate regions of the parameter space. When a calculation is undertaken taking into account the actual angle of orientation of Lake Michigan the inviscid theory predicts a direction of rotation of the nodal lines which is opposite to that observed by Mortimer & Fee (1976). Therefore, an investigation of the effect of friction on the generation of tides in an enclosed basin was conducted in the event that this modification of the theory might account for the observed response of Lake Michigan.

5. The model in the presence of bottom friction

(a) *Formulation of the model*

Although friction is more realistically modelled by a square law, for analytical convenience the frictional forces are accounted for by a linear dependence on the velocity component. This approximation is considered valid at least for the purpose of determining the general behaviour of the solutions under the influence of bottom friction.

As a result of the assumption of the linear frictional law, the dimensional bottom stress components in the local coordinate system are of the form

$$F_x B = \rho h K \sigma \mu; \quad F_y B = \rho h K \sigma v.$$

The Laplace tidal equations become, in the local coordinate system and in non-dimensional form,

$$\frac{\partial u}{\partial t} - v = -\epsilon \frac{\partial \eta}{\partial x} - K\omega u + \epsilon F_x,$$

$$\frac{\partial v}{\partial t} + u = -\epsilon \frac{\partial \eta}{\partial y} - K\omega v + \epsilon F_y,$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0,$$

where K is an unspecified coefficient of friction and F_x and F_y each denote the real part of a complex quantity.

$$F_x = R[(F'_y \sin\gamma - iF'_x \cos\gamma)\, e^{i\omega t}],$$

and

$$F_y = R[(F'_y \cos\gamma + iF'_x \sin\gamma)\, e^{i\omega t}],$$

where F'_x and F'_y refer to the non-dimensional amplitudes of tidal generating forces in the global coordinate system. Assuming a solution of the form:

$$u = R(U e^{i\omega t}),$$

$$v = R(V e^{i\omega t}),$$

and

$$\eta = R(Ax\, e^{i\omega t} + By\, e^{i\omega t}),$$

the complex components U, V, A and B are found by substitution as before in the modified Laplace tidal equations.

Once the algebra is performed, the solutions are as follows

$$U = \left\{\left[i\omega + K\omega + \frac{(1+a)\epsilon}{i\omega}\right]\epsilon F_x + \epsilon F_y\right\} \Big/ \left\{1 + \left[i\omega + K\omega + \frac{(1-a)\epsilon}{i\omega}\right]\left[i\omega + K\omega + \frac{(1+a)\epsilon}{i\omega}\right]\right\}, \tag{11}$$

$$V = \left\{-\epsilon F_x + \epsilon F_y \left[i\omega + K\omega + \frac{(1-a)\epsilon}{i\omega}\right]\right\} \Big/ \left\{1 + \left[(i\omega + K\omega) + \frac{(1-a)\epsilon}{i\omega}\right]\left[i\omega + K\omega + \frac{(1+a)\epsilon}{i\omega}\right]\right\}. \tag{12}$$

A and B are expressed by relations similar to equations (5) and (6).

(b) *Discussion*

In the hope that the modification due to bottom friction will shed some light on the anomalous behaviour of Lake Michigan, the discussion will be focused upon the direction of rotation of the phase response.

In contrast to the inviscid model where the sense of the response was simply determined by settling U and V to zero we have a considerably more complicated condition of zero rotation in the frictional model.

$$UV^* = U^*V, \tag{13}$$

where the asterisk symbolizes the complex conjugate. When relations (11) and (12) are substituted into condition (13) the relatively simple result follows for the case $\gamma = 0, \pm\frac{1}{2}\pi$ (figure 4)

$$\epsilon^2(1-a^2)F_xF_y + [-2F_xF_ya\omega^2 - F_x^2\omega(1+a) - F_y^2\omega(1-a)]\epsilon + (F_x^2+F_y^2)\omega^3 + F_xF_y(\omega^2+\omega^4) + K^2\omega^4 = 0.$$

(For north–south alinement the roles of F_x and F_y are reversed.) If this equation is considered as a quadratic relation in the variable ϵ, it is evident that the effect of friction increases the product of the roots while the sum remains unaltered. Therefore, the larger root is decreased by friction while the smaller root is increased which implies for the case of north–south or east–west orientation of the major axis of the lake, friction has a tendency to decrease the region of inverted response. Stated in terms of the semidiurnal case, friction tends to increase the tendency for clockwise rotation of the tidal response.

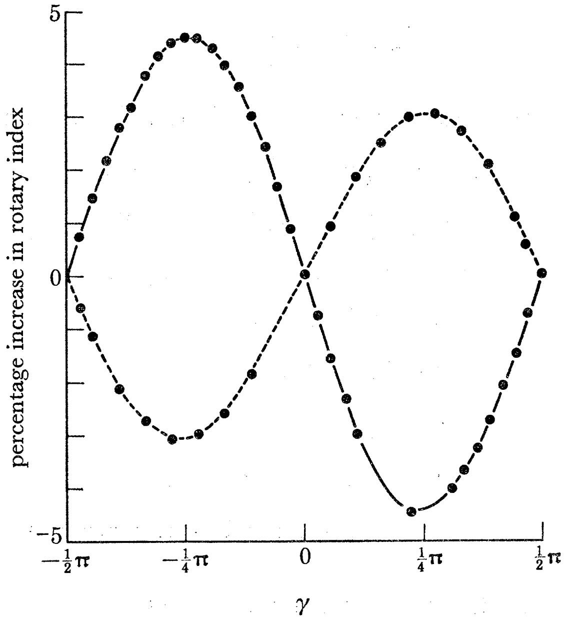

FIGURE 4. Percentage increase in rotary index, for the friction coefficient, K, of 0.5 as a function of the angle of orientation, γ, of the major axis for a basin of $a = 0.9$, $\epsilon = 150$, $\theta = 40°$ N for the semidiurnal (——) and diurnal (------) response.

On the other hand, an interesting situation arises for axis orientation at intermediate angles, for example, at $\frac{1}{4}m\pi$ ($m = \pm 1$). For these angles relation (13) may be expressed once more as a quadratic polynomial in ϵ,

$$2F_xF_y(1-a^2)\epsilon^2 + [2\omega(F_x^2+F_y^2) - 4\omega^2F_{xy} - 2m(F_x^2-F_y^2)Ka\omega^2]\epsilon + 2\omega^3(F_x^2+F_y^2) + 2F_xF_y(\omega^4+K\omega^2+\omega^2) = 0.$$

Not only is the product of the roots increased as discussed above, but also the sum is increased for the semidiurnal oscillation ($F_x > F_y$) when the orientation is positive and sum is decreased when the orientation is negative. If the sum of the roots increases, the larger root increases while the lesser root decreases. Thus, for positive orientation at angles not $\frac{1}{2}\pi$ the consequence of bottom friction is to enhance the tendency for counterclockwise response of the semidiurnal oscillation. Figure 4 demonstrates this effect for the diurnal and semidiurnal responses at 40° N for a basin of ϵ, a coordinates, (150, 0.9). The percentage change in a normalized version of equation (13), defined as the rotary index

$$-\mathrm{i}(UV^* - U^*V)/(UU^* + VV^*)$$

at a friction coefficient of 0.5 is plotted against angle of orientation. Friction has its most pronounced influence on the tendency to cyclonic response at a basin orientation of 45°.

Other more obvious aspects of frictional influence such as the effect of friction on the amplitude response are noted in the following section on the application of the concepts discussed to natural basins.

6. Application of the theory to tides in natural basins

In order to compare the theoretical predictions with observational evidence, seven basins of dimensions appropriate to the assumptions upon which the theory is based were selected. There exists at least qualitative evidence of the tidal response if not quantitative in each of the basins studied.

Without sufficient bathymetric data to construct an objective analysis of the appropriate geometrical parameters of the basin by, for example, least square techniques, a problem exists of how to choose L, H, W. The problem has been solved rather arbitrarily by choosing a model elliptical paraboloid such that its surface area and volume are identical to the prototype. As a third relation the model basin is specified in such a way that its longitudinal period of oscillation is the same as that of the natural basin. The observed periods of free oscillation and the derived geometrical factors are summarized in table 1.

For the sake of comparison of theory with the field data the following quantities were calculated from the solutions for each basin, the semi-major and semi-minor axis of the constituent tidal

Table 2. Theoretical constituents of the semidiurnal tidal response for seven enclosed basins with and without bottom friction

basin	K	velocity semi-major axis (mm/s)	velocity semi-minor axis (10^{-2} mm/s)	angle of orientation (deg)	phase angle (deg)	free surface amplitude (cm)	rotary index
Superior	0	2.5	−2	0	0	3.0	0.0120
	0.5	2.3	−1	−0.4	−18.9	2.8	0.0122
Michigan	0	3.9	−1	−0.31	44.2	3.1	0.005
	0.5	3.2	—	−0.83	12.0	2.6	−0.002
Huron	0	1.2	−8	2.21	−39.1	1.2	0.1509
	0.5	1.1	−9	1.17	−50	1.1	0.164
Erie	0	14.9	49	0.15	−20.6	3.7	−0.067
	0.5	6.5	25	−0.75	85.0	2.4	−0.077
Ontario	0	0.75	−2	−0.2	6.9	1.09	0.042
	0.5	0.75	−2	−0.34	1.2	1.08	0.041
Baikal	0	0.52	−0.1	−0.02	43.6	2.3	0.00285
	0.5	0.51	—	−0.03	38.4	2.3	0.00279
Black Sea	0	0.75	−5.4	0	0	4.0	0.1426
	0.5	0.74	−5.3	−10.45	−5.6	4.0	0.1427

ellipse (Godin 1972) in dimensional form, the angle of orientation of the major axis of the constituent ellipse with respect to the east direction, the phase angle, the rotary index and the amplitude of the deviation of the free surface at the distal points on the major axis of the basin. These quantities were computed both for the inviscid model and for the frictional model with a coefficient of friction of 0.5. Theoretical response characteristics are derived for the semidiurnal tide in table 2 and for the diurnal tide in table 3.

Table 3. Theoretical constituents of the diurnal tidal response for seven enclosed basins with and without bottom friction

basin	K	velocity semi-major axis (mm/s)	velocity semi-minor axis (10^{-2} mm/s)	angle of orientation (deg)	phase angle (deg)	free surface amplitude (cm)	rotary index
Superior	0	0.54	0.2	0	0	1.2	−0.004831
	0.5	0.54	0.2	−0.02	−3.46	1.2	−0.004830
Michigan	0	0.42	0.5	−2.1	−6.7	0.6	−0.024
	0.5	0.42	0.6	−2.1	−11.6	0.6	−0.031
Huron	0	0.22	0.5	8.6	2.4	0.41	−0.047
	0.5	0.22	0.5	8.6	0.24	0.41	−0.037
Erie	0	1.9	0.3	−0.59	−5.6	0.9	−0.034
	0.5	1.8	0.3	−0.54	−20.4	0.85	−0.039
Ontario	0	0.19	0.2	−0.43	−0.74	0.6	−0.011
	0.5	0.19	0.2	−0.42	−2.1	0.6	−0.012
Baikal	0	0.14	−0.1	−0.1	22.5	1.2	0.00242
	0.5	0.14	−0.1	−0.1	21.1	1.2	0.00235
Black Sea	0	0.19	0.4	0	0	2	−0.0402
	0.5	0.19	0.4	0.01	−1.3	2	−0.0402

As a consequence of the lack of known studies of the tides of Lake Huron an analysis of the water level data discussed by Freeman & Murty (1972) was undertaken by the method of cross-spectral analysis. The phase between components of the variance lying in the 12.4 h band revealed that a station on the eastern shoreline, Goderich, lags the station at the southern end by 23° which in turn lags a station on the western shore, Harbor Beach, by 9°. The predicted clockwise sense of the semidiurnal response is corroborated.

The tides of the Black Sea are discussed at length by Defant (1961). The predicted amplitude is somewhat larger for both the M_2 and K_1 tides at a station at the eastern extremity of the basin, although on the west coast the amplitude of the M_2 is in close agreement. The direction of the semidiurnal amphidromy is clockwise while the direction of the K_1 is unknown from Defant's data.

Defant also presents observational and theoretical data for the semidiurnal and diurnal tides of Lake Baikal. Predicted amplitude of the K_1 component is slightly larger than the M_2 which is a consequence of the extreme elongation of the lake (see equation (5)). However, the observed amplitude, K_1, is somewhat smaller than the amplitude of M_2 tide. Moreover, the theory overestimates the amplitude, 2.3 cm, as opposed to an observed amplitude of 1.6 cm. Agreement between theory and observation cannot be reached by invoking friction of the type studied here. The predicted sense of rotation of the M_2 component is in agreement with a theoretical response presented by Defant.

The short-period tides of Lake Erie have been extensively investigated by Platzman (1966). The spectral signature of M_2 constituent at the western end of the lake corresponds to an amplitude of 2.7 cm which falls between the predicted values on table 2. A coefficient of friction of $K = 0.2$

yields an exact agreement. On the other hand, the observed amplitude of the spectral peak at 24 h is too large to be attributed to the gravitational tide. Platzman suggests that these peaks must be a 'meteorological tide'. By means of harmonic analysis of the water levels of 4 stations around the lake, Platzman determined the phase of the semidiurnal (S_2) tide which is considered to respond in a similar manner to the M_2 constituent. These results are consistent with the predicted counterclockwise response of the lake.

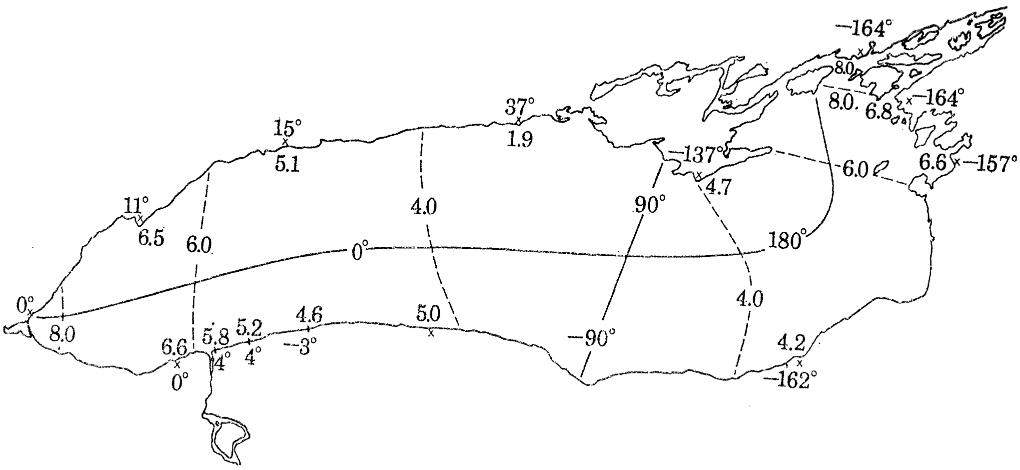

Figure 5. Phase angle (——, degrees) and amplitude (----, mm) response of the semidiurnal tide of Lake Ontario. Estimates are based on an analysis of water level data collected in summer and fall of 1972. Positive angles are associated with phase lag.

The 12.4 h gravitational tide of Lake Ontario has been studied by means of cross-spectral analysis as outlined by Hamblin (1974) of water level readings collected from 12 stations during an intensive observational programme known as the International Field Year for the Great Lakes. The predicted amplitudes at the two extremities of the lake are evidently 25 % larger than observed for both the frictional and frictionless models. The direction of rotation of the free surface is in agreement with the predicted response, see figure 5.

Observations of the semidiurnal tide in the largest of the North American inland seas, Lake Superior, Mortimer & Fee (1976) have established the existence of the clockwise amphidromy which is in agreement with the predictions. The amplitude of this component at Duluth in the western limit of the basin is 2.0 cm (Defant 1961) which is 50 % smaller than the predicted inviscid case and 90 % less than the viscid model. Because of the problems in distinguishing the gravitational tidal response from that of the meteorological tide no attempt is made here to compare the diurnal amplitude with Defant's data.

Finally, the most remarkable observed feature is the well established presence of the counterclockwise semidiurnal response of Lake Michigan, Mortimer & Fee (1976). The sign of the rotary index is a useful qualitative indication of the direction of rotation. Figure 6 shows the dependence of this index on the coefficient of friction for the case of Lake Michigan. It is evident that for coefficients greater than 0.37 the response is in agreement with Mortimer & Fee's findings. Pekeris & Accad (1969) indicate from their investigations that, where frictional effects are of importance, K may reach a value as high as 0.5 for the M_2 tide. It is likely that the relatively high coefficients of tidal friction encountered in tidal streams in the English Channel (Channon & Hamilton 1971) do not apply in the case of a closed basin where owing to the low speeds the bottom friction is probably hydrodynamically smooth. On the other hand, the coastline and

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bathymetry are not smooth in general. Qualitatively, the dissipation of energy due to the eddies formed by sharp promontories or irregularities may result in relatively high bottom friction.

It is interesting that as discussed earlier there is not a general tendency for the rotary index to be decreased by friction but only within a certain range of orientation of the major axis of the basin. By geographical coincidence Lake Michigan is situated in such a manner that the observed response can be attributed to the effects of friction. Other possible explanations, such as the fact that Lake Michigan is connected to Lake Huron through a narrow passage and the effects of higher gravitational modes excited by the tidal forces, must await assessment by means of numerical computations.

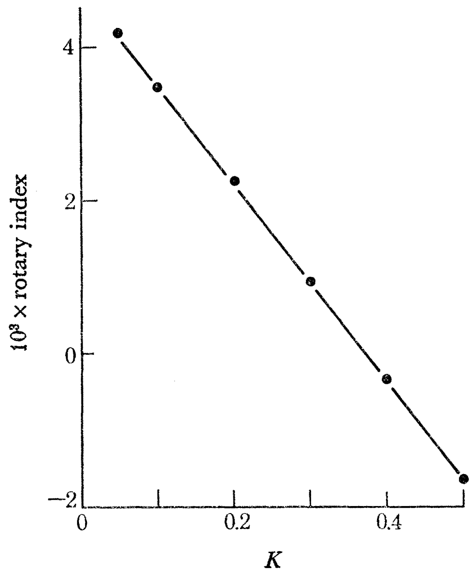

Figure 6. Rotary index as a function of friction coefficient for Lake Michigan.

7. Summary

An analytical model in which the free surface assumes a simple planar form amphidromic about the origin is employed to explore the oscillatory response to the short-period astronomical tidal forces. It has been demonstrated that the tidal response may be decomposed into three modes, a quasi-static forced mode, a longitudinal free mode and a transverse free mode. The character of the response depends on which of the modes is excited most prominently. At periods of forcing much greater than the natural periods of oscillation the solution is forced in a direct manner and rotates about the origin in the direction of the forcing function. If the period of forcing coincides approximately with the period of the longitudinal seiche, a standing Kelvin wave is perturbed which is associated with a cyclonic rotation of the nodal lines. The mode of the third kind is similar to a standing Poincaré wave since there is an associated anticyclonic progression of phase. This type of response is generated when the period of forcing is in the vicinity of the transverse period of the free oscillation.

Validation of the predicted features of the tidal response is limited by the availability of empirical evidence. Platzman's (1966) careful analysis of the daily fluctuations in the water levels of an inland sea suggests that the observed diurnal response is due to local effects such as the oscillating force of the land-lake–breeze system. Fortunately, the signature of the semidiurnal tide is apparently contaminated by local influences to a lesser degree. Knowledge of the behaviour of the 12.4 h tide gained both from published studies and analysis undertaken here have confirmed at least the qualitative predictions of the theory.

The progression of phase around the basin can be readily determined by either harmonic analysis or by cross-spectral analysis of as few as two pairs of water level gauges on the periphery of the basin. The direction of advance of these is compared to the predicted character of the response which is indicated by a parameter derived from the solution known as the rotary index. In all cases with the exception of Lake Michigan the signs of the rotary index for both the frictionless and viscid model are in agreement with observations.

The response of Lake Erie and Lake Michigan is of the longitudinal or Kelvin wave type although interestingly, for Lake Michigan in contrast to Lake Erie, the 12.4 h period of forcing is greater than both periods of free oscillation. It is argued that a reasonable magnitude of bottom friction in combination with an appropriate alinement of the major axis of the basin can account for the anomalous behaviour of Lake Michigan.

In all basins where the amplitude response is known, the amplitude is overestimated by the theory. Only in the case of Lake Erie is it possible to account for this discrepancy by reasonable values of the friction coefficient. The effect of friction on the amplitude response is most pronounced in basins which exist in a state of near resonance between the period of forcing and the natural periods of oscillation.

The author wishes to acknowledge the cooperation of the United States Lake Survey, N.O.A.A. and the Marine Sciences Directorate of the Department of Environment who have kindly supplied water level readings on Lake Ontario. Discussions with N. S. Heaps who first suggested the problem have been most helpful. The author is grateful to D. B. Rao for a number of improvements.

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